

# OVER RECURRENCE FOR MIXING TRANSFORMATIONS

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**ABSTRACT.** We show that every invertible strong mixing transformation on a Lebesgue space has strictly over-recurrent sets. Also, we give an explicit procedure for constructing strong mixing transformations with no under-recurrent sets. This answers both parts of the first question posed in [2].

We define  $\epsilon$ -over-recurrence and show that given  $\epsilon > 0$ , any ergodic measure preserving invertible transformation (including discrete spectrum) has  $\epsilon$ -over-recurrent sets of arbitrarily small measure. Discrete spectrum transformations and rotations do not have over-recurrent sets, but we construct a weak mixing rigid transformation with strictly over-recurrent sets.

## 1. INTRODUCTION

We answer a two-part question posed in [2]. It is the first question raised in [3] on page 50.

**Question 1.1.** *“Is it true that for any invertible mixing measure preserving system  $(X, \mathcal{B}, \mu, T)$  there exists  $A \in \mathcal{B}$  with  $\mu(A) > 0$  such that for all  $n \neq 0$ ,  $\mu(A \cap T^n A) < \mu(A)^2$ ? How about the reverse inequality  $\mu(A \cap T^n A) > \mu(A)^2$ ?”*

The answer is different for each part. Respectively, the answers are “no”, and “yes”. One of the key differences is a basic lemma on set intersections which is presented in Lemma 4.2. However, this lemma alone is not sufficient, and in particular, this lemma is pointed out in [3]. In the next section, we prove that the answer to the second part is “yes”. This is done by constructing a set of positive measure such that the given mixing transformation mixes the set slowly. This answers the same question raised in [4].

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The notions of over-recurrent, under-recurrent, strictly over-recurrent and strictly under-recurrent are defined in [4] as a means for addressing Question 1.1 and similar questions. We expand these definitions to include the weaker notion of  $\epsilon$ -over-recurrent and  $\epsilon$ -under-recurrent. See section 2 for definitions. It is straight-forward to show that any partially rigid transformation has no under-recurrent set. The class of partially rigid transformations is larger than the class of rigid transformations and was first introduced by N. Friedman in [6]. As a preliminary result, we give a short proof that any discrete spectrum transformation (or rotation) does not have an over-recurrent set. However, we show that given an invertible ergodic measure preserving transformation  $T$  and  $\epsilon > 0$ ,  $T$  has  $\epsilon$ -over-recurrent sets with arbitrarily small measure. Also, we construct a rigid weak mixing transformation that has a strictly over-recurrent set. The question of whether every weak mixing transformation has an over-recurrent set remains open.

To answer the first part of Question 1.1, we give a general procedure for constructing a strong mixing transformation from an input mixing transformation and an arbitrary rigid transformation. We gradually diminish the effects of the rigid transformation, but in the process, build a strong mixing transformation that acts like a rigid transformation on a shrinking part of the measure space. We use a technique from [1] to produce this slow mixing transformation.

Finally, in the last section, we point out that the same construction for producing a slow mixing transformation can be used to construct a strong mixing transformation with singular spectrum from any strong mixing transformation. Thus, any strong mixing transformation can be multiplexed with any rigid transformation to produce a transformation that is mixing of all orders.

## 2. PRELIMINARIES

All transformations are assumed to be invertible, ergodic and measure preserving on a fixed Lebesgue probability space  $(X, \mathcal{B}, \mu)$ , and all sets are assumed to be measurable. Let  $\mathbb{N} = \{1, 2, \dots\}$  be the natural numbers and  $\mathbb{Z}$  the set of integers. The following definitions are expanded from [4].

**Definition 2.1.** *Let  $A$  be a measurable set such that  $0 < \mu(A) < 1$ .*

- (1) *Set  $A$  is over-recurrent if  $\mu(T^n A \cap A) \geq \mu(A)^2$  for  $n \in \mathbb{Z}$ .*
- (2) *Set  $A$  is under-recurrent if  $\mu(T^n A \cap A) \leq \mu(A)^2$  for  $n \in \mathbb{N}$ .*
- (3) *Set  $A$  is strictly over-recurrent if  $\mu(T^n A \cap A) > \mu(A)^2$  for  $n \in \mathbb{Z}$ .*
- (4) *Set  $A$  is strictly under-recurrent if  $\mu(T^n A \cap A) < \mu(A)^2$  for  $n \in \mathbb{N}$ .*
- (5) *Set  $A$  is  $\epsilon$ -over-recurrent if  $\mu(T^n A \cap A) > (1 - \epsilon)\mu(A)^2$  for  $n \in \mathbb{Z}$ .*
- (6) *Set  $A$  is  $\epsilon$ -under-recurrent if  $\mu(T^n A \cap A) < (1 + \epsilon)\mu(A)^2$  for  $n \in \mathbb{N}$ .*

These definitions are motivated by the Khintchine recurrence theorem [9]. It was shown in [4] that there exist mixing transformations with no under-recurrent sets. Also, under-recurrent functions are defined, and it is shown that any transformation with singular maximal spectral type has no under-recurrent function. While we give a general construction of mixing transformations with no under-recurrent sets, our main results concern (strictly) over-recurrent sets. All results were obtained independently of [4].

### 3. OVER-RECURRENT SETS

This section focuses on results related to (strictly) over-recurrent sets. First, we prove that any strong mixing transformation has a strictly over-recurrent set.

**Theorem 3.1.** *Let  $T$  be an invertible mixing transformation on a Lebesgue probability space. Then  $T$  has strictly over-recurrent sets  $A$  of arbitrarily small measure. In particular,  $\mu(T^n A \cap A) > \mu(A)^2$  for all  $n \in \mathbb{Z}$ .*

**Proof:** Let  $a \in \mathbb{R}$  be such that  $0 < a < \frac{1}{4}$  and let  $a_i = \frac{a}{i(i+1)}$  for  $i \in \mathbb{N}$ . Note that

$$\sum_{i=1}^{\infty} a_i = a \sum_{i=1}^{\infty} \left( \frac{1}{i} - \frac{1}{i+1} \right) = a.$$

For  $j \in \mathbb{N}$ , choose  $\epsilon_j > 0$  such that

$$(1) \quad (1 - \epsilon_j) \left( 1 + \frac{j(1 - 2a) + (1 - 4a)}{(j+1)(j+2)} \right) > 1.$$

We will define an infinite sequence  $A_i$  of disjoint measurable sets such that  $\mu(A_i) = a_i$  for  $i \in \mathbb{N}$ , and  $A = \bigcup_{i=1}^{\infty} A_i$ . Let  $A_1$  be any set with measure  $a/2$ . Since  $T$  is mixing, there exists  $N_1 \in \mathbb{N}$  such that for  $|n| \geq N_1$ ,

$$|\mu(T^n A_1 \cap A_1) - \mu(A_1)^2| < \epsilon_1 \mu(A_1)^2.$$

Choose  $m_1 \in \mathbb{N}$  such that  $m_1 > \frac{1}{\epsilon_1}$  and most points have a good ergodic average for  $A_1$  of length  $m_1 N_1$ . Let  $B_1$  be the base of a Rohklin tower of height  $m_1 N_1$  such that

$$\mu\left(\bigcup_{i=0}^{m_1 N_1 - 1} T^i(B_1)\right) > 1 - \epsilon_1.$$

Choose a subset  $I_1 \subset B_1$  such that the set

$$A_2 = \{T^i x : x \in I_1, 0 \leq i < m_1 N_1, T^i x \notin A_1\}$$

satisfies  $\mu(A_2) = a_2$ . Note that for  $i \in \mathbb{N}$  such that  $|i| < N_1$ ,

$$\mu(T^i A_2 \cap A) > (1 - \epsilon_1) \mu(A_2).$$

We repeat this inductively. Given  $A_1, A_2, \dots, A_k$ , choose  $A_{k+1}$  in the following manner. Let  $C_k = \bigcup_{i=1}^k A_i$ . Since  $T$  is mixing, there exists  $N_k > N_{k-1}$  such that for  $|n| \geq N_k$ ,

$$|\mu(T^n C_k \cap C_k) - \mu(C_k)^2| < \epsilon_k \mu(C_k)^2.$$

Choose  $m_k \in \mathbb{N}$  such that  $m_k > \frac{1}{\epsilon_k}$  and most points have a good ergodic average for  $C_k$  of length  $m_k N_k$ . Let  $B_k$  be the base of a Rohklin tower of height  $m_k N_k$  such that

$$\mu\left(\bigcup_{i=0}^{m_k N_k - 1} T^i(B_k)\right) > 1 - \epsilon_k.$$

Choose a subset  $I_k \subset B_k$  such that the set

$$A_{k+1} = \{T^i x : x \in I_k, 0 \leq i < m_k N_k, T^i x \notin C_k\}$$

satisfies  $\mu(A_{k+1}) = a_{k+1}$ . The set  $A_{k+1}$  satisfies the property that for  $i \in \mathbb{N}$  such that  $|i| < N_k$ ,

$$\mu(T^i A_{k+1} \cap A) > (1 - \epsilon_k) \mu(A_{k+1}).$$

Now, we show the set  $A = \bigcup_{i=1}^{\infty} A_i$  is over-recurrent. If  $n$  is a natural number such that  $|n| < N_1$ , then

$$\begin{aligned} \mu(T^n(\bigcup_{k=2}^{\infty} A_k) \cap A) &> (1 - \epsilon_1) \mu(\bigcup_{k=2}^{\infty} A_k) = \frac{1}{2}(1 - \epsilon_1)a \\ &> \frac{1}{2}\left(1 - \frac{2 - 6a}{6}\right)a = \left(\frac{1}{3} + \frac{a}{2}\right)a > a^2. \end{aligned}$$

Let  $k \in \mathbb{N}$ , and  $n \in \mathbb{N}$  be such that  $N_k \leq n < N_{k+1}$ . The set  $A$  is a disjoint union of the following three sets:  $C_k, A_{k+1}, \bigcup_{i=k+2}^{\infty} A_i$ . For convenience, set  $C_{k,1} = C_k$ ,  $C_{k,2} = A_{k+1}$  and  $C_{k,3} = \bigcup_{i=k+2}^{\infty} A_i$ . Thus,

$$\begin{aligned} \mu(T^n A \cap A) &= \sum_{i=1}^3 \mu(T^n C_{k,i} \cap C_{k,i}) + \sum_{i=2}^3 \mu(T^n C_{k,i} \cap A) \\ &> (1 - \epsilon_k) \mu(C_{k,1})^2 + (1 - \epsilon_k) \mu(C_{k,3}) \\ &= (1 - \epsilon_k) \left( \left(a - \frac{a}{k+1}\right)^2 + \frac{a}{k+2} \right) \\ &> (1 - \epsilon_k) a^2 \left( 1 + \frac{k(1 - 2a) + 1 - 4a}{a(k+1)(k+2)} \right) \\ &> a^2, \quad \text{by 1.} \quad \square \end{aligned}$$

Note, the method used for choosing  $A_i$  can be used to show that mixing transformations have no uniform rate over all measurable sets. Given any

sequence  $\delta_i \rightarrow 0$ , there exist parameters  $\epsilon_i, N_i, m_i$  and  $A_i$  such that

$$\lim_{n \rightarrow \infty} \frac{\delta_n}{\mu(T^n A \cap A) - \mu(A)^2} = 0.$$

This is already well known to be true, and follows from a general argument of Krengel [8] on the lack of uniform rates for the ergodic theorem.

**3.1. Over-recurrence for non-mixing transformations.** The previous result can be used to construct a rigid weak mixing transformation that has a strictly over-recurrent set. First, we show that any discrete spectrum transformation does not have an over-recurrent set.

**Proposition 3.2.** *If  $T$  has discrete spectrum, then  $T$  has no over-recurrent sets.*

**Proof:** Let  $A$  be any measurable set such that  $0 < \mu(A) < 1$ . Since  $T$  has discrete spectrum, there exist a sequence of refining towers of heights  $h_n$  and integers  $k_n \geq 2$  such that  $h_{n+1} = k_n h_n$ . Choose  $m$  such that the tower of height  $h_m$  has a union  $J$  of levels that approximates  $A$ . In particular, choose  $\delta$  and  $m$  such that  $2\delta < 1 - \mu(A)$  and

$$\mu(A \triangle J) < \frac{\delta \mu(A)}{4}.$$

Note that  $\mu(T^{ih_m} J \cap J) = \mu(J)$  for all  $i \in \mathbb{Z}$ . Thus,  $\mu(T^{ih_m} A \cap A) > (1 - \delta)\mu(A) > \mu(A)^2$  for all  $i \in \mathbb{Z}$ . By the  $L_2$  ergodic theorem,

$$\lim_{n \rightarrow \infty} \frac{1}{h_n} \sum_{i=0}^{h_n-1} \mu(T^i A \cap A) = \mu(A)^2.$$

Since  $\{ih_m | i \in \mathbb{N}\}$  forms a subsequence of positive density in  $\mathbb{N}$ , then there must exist  $i$  and  $j$  such that  $0 < j < h_m$  and

$$\mu(T^{ih_m+j} A \cap A) < \mu(A)^2. \quad \square$$

A similar argument can be used to show that ergodic rotations do not have over-recurrent sets.

The following may seem a bit surprising intuitively, but it is not difficult to prove.

**Proposition 3.3.** *Given any invertible ergodic measure preserving transformation  $T$  and  $\epsilon > 0$ ,  $T$  has  $\epsilon$ -over-recurrent sets of arbitrarily small measure.*

**Proof:** Let  $\epsilon, \delta > 0$ . Let  $S$  be a rank-one strong mixing transformation. By Theorem 3.1,  $S$  has a strictly over-recurrent set  $A$  with measure less than  $\delta$ . Choose a tower for  $S$  of height  $h$  and a union  $J$  of levels from the tower

that approximate  $A$  well, and such that the complement of the tower has measure less than  $\epsilon\mu(A)/4$ . Also, assume

$$\mu(A \triangle J) < \frac{\epsilon}{4}\mu(A).$$

Choose a Rokhlin tower for  $T$  of height  $h$  such that the complement of the tower has measure less than  $\epsilon\mu(A)/4h$ . There is a one-to-one onto correspondence between the levels of the  $T$  tower and the levels of the  $S$  tower. Take the correspondence that preserves the order of the levels from top to bottom of the towers. The set  $J$  in the  $S$  tower matches a set  $K$  in the  $T$  tower. It is not difficult to prove that the set  $K$  is  $\epsilon$ -over-recurrent for the transformation  $T$ .  $\square$

**3.2. Rigid weak mixing transformations with strictly over-recurrent sets.** We prove there exist rigid weak mixing transformations with strictly over-recurrent sets.

**Theorem 3.4.** *There exist rigid weak mixing transformations  $T$  and sets  $A$  such that for all  $n \in \mathbb{Z}$ ,*

$$\mu(T^n A \cap A) > \mu(A)^2.$$

**Proof:** Let  $S_1$  be a rank-one mixing transformation such as Ornstein's mixing rank-one, or the (Adams-Smorodinsky) staircase transformation. By Theorem 3.1, there is an over-recurrent set  $A_1$  of arbitrarily small measure. By the technique used in Proposition 3.3, there exists a tower of height  $h$  such that the set  $A$  is  $\epsilon/4$ -over-recurrent, even if we modify  $S$  to be discrete spectrum from this point on. Similarly, we can cut this tower into  $r_1$  sub-columns of equal width, and stack to produce a rigid time (as  $r_n \rightarrow \infty$ ). Resume the definition of the mixing transformation  $S_2$  similar to  $S_1$ . Then define a set  $A_2$  as in Theorem 3.1, such that iterates of  $A_2$  overlap itself for a long time (forward and backward in time). Since  $S_2$  is mixing, it will mix  $A_1 \cup A_2$  over time. Once this happens sufficiently well, then introduce another rigid time  $r_2$ . It will not disturb the near over-recurrence of  $A_1 \cup A_2$ . The error in the near over-recurrence can be forced to be much smaller than the size of set  $A_3$ . The set  $A_3$  is defined to be nearly fixed for a long period of time compared to the last mixing times chosen for  $S_2$ , as it operates on  $A_1 \cup A_2$ .

This is repeated inductively to produce a rigid weak mixing transformation. The arguments used in Theorem 3.1 and Proposition 3.3 can be applied here to show that the set  $A = \bigcup_{i=1}^{\infty} A_i$  is strictly over-recurrent for the resulting transformation  $T$ . The transformation  $T$  may be defined as

$$T = \lim_{n \rightarrow \infty} S_n.$$

Although, each  $S_n$  may be strong mixing, the limiting transformation  $T$  will be rigid weak mixing, if  $r_n \rightarrow \infty$ .  $\square$

#### 4. SLOW STRONG MIXING TRANSFORMATIONS

In this section, we prove the following theorem.

**Theorem 4.1.** *There exists a strong mixing transformation  $T$  such that for every set  $A$  satisfying  $0 < \mu(A) < 1$ , the following set is infinite:*

$$\{n \in \mathbb{N} : \mu(T^n A \cap A) - \mu(A)^2 > 0\}.$$

We use a technique from [1] to construct our example. In [1], a method is given for combining two transformations to produce a third "multiplexed" transformation. In that paper, the two input transformations are a rigid ergodic transformation and a weak mixing transformation. The output transformation is a rigid weak mixing transformation. In this case, our input transformations are a strong mixing transformation and a rigid transformation. The output is a strong mixing transformation.

We use the following standard result from measure theory.

**Lemma 4.2.** *Let  $(X, \mu)$  be a probability space. Given  $\epsilon > 0$  and  $0 < \alpha \leq 1$ , there exists  $N$  such that for any measurable sets  $A_1, A_2, \dots, A_N$  satisfying  $\mu(A_i) = \alpha$  for  $1 \leq i \leq N$ , there exist  $1 \leq j < k \leq N$  such that*

$$\mu(A_j \cap A_k) > \alpha^2 - \epsilon.$$

**Proof:**

$$(2) \quad \int \left( \sum_{i=1}^N I_{A_i} \right)^2 d\mu = \sum_{i \neq j} \mu(A_i \cap A_j) + \sum_{i=1}^N \mu(A_i)$$

$$(3) \quad \geq \left( \sum_{i=1}^N \mu(A_i) \right)^2 = N^2 \alpha^2$$

Therefore,

$$(4) \quad \frac{1}{N^2} \sum_{i \neq j} \mu(A_i \cap A_j) \geq \alpha^2 - \frac{\alpha}{N}$$

and we have our result.  $\square$

**4.1. Mixing Counterexample.** The towerplex method was first defined in section 2 of [1]. The roles of the input transformations are different. In this case, we use  $S$  to represent the first input transformation which will be a strongly mixing transformation. The second input transformation will be a rigid transformation denoted by  $R$ . Thus, a sequence of transformations  $S_n : Y_n \rightarrow Y_n$  will be defined such that  $S_n$  is isomorphic to  $S$ , and another

sequence  $R_n : X_n \rightarrow X_n$  such that  $R_n$  is isomorphic to  $R$ . For each  $n \in \mathbb{N}$ ,  $X_n \cup Y_n = X$  and define

$$T_n(x) = \begin{cases} R_n(x) & \text{if } x \in X_n \\ S_n(x) & \text{if } x \in Y_n. \end{cases}$$

Then the output transformation is defined by  $T(x) = \lim_{n \rightarrow \infty} T_n(x)$  for  $x \in X$ .

Two main parameters are used to control the properties of  $T$ :

$$(5) \quad s_n = \frac{1}{2(n+2)} \quad \text{and} \quad r_n = \frac{1}{2}.$$

The parameter  $s_n$  represents the proportion of mass that transfers from  $Y_n$  to  $X_n$  at each stage. Similarly,  $r_n$  represents the proportion of mass that transfers from  $X_n$  to  $Y_n$  at each stage. These settings cause  $\lim_{n \rightarrow \infty} \mu(Y_n) = 1$  and consequently  $\lim_{n \rightarrow \infty} \mu(X_n) = 0$ . Note, the fact that  $S_n$  is mixing is not sufficient to prove that  $T$  is mixing. On the other hand, the fact that  $T_n$  is not ergodic does not prevent  $T$  from being ergodic. We are more careful about defining  $S_{n+1}$  based on  $S_n$  and use a property called "isomorphism chain consistency" to show that  $T$  is strongly mixing.

This provides a general technique for constructing a slow strong mixing transformation from an arbitrary strong mixing transformation and an arbitrary rigid transformation. As in [1], we can have for each  $n \in \mathbb{N}$ ,  $1/(n+2) < \mu(X_n) < 1/n$  and  $(n+1)/(n+2) > \mu(Y_n) > (n-1)/n$ . Also, since the measure of  $X_n$  goes to zero slow enough, and the  $X_n$  are approximately independent, then  $X_n$  will mix with any measurable set. Also,  $R_n$  rigid on  $X_n$  will cause  $T_n$  to be approximately  $1/(n+1)$  rigid on  $X$ . In this way, we can slow the rate of mixing, because  $T$  will resemble a rigid transformation for arbitrarily long times on  $X_n$ .

**4.2. Slow mixing from dissipating rigidity.** Lemma 4.2 will inform us on how long  $S_n$  should run before phasing in  $S_{n+1}$  to guarantee the intersection

$$(6) \quad \mu(S_n^i A \cap A) > (1 - \frac{\delta}{n})\mu(A)^2$$

for some  $i$  and  $\delta$ . Let  $N$  be large enough to guarantee (6) holds for some  $i$  from a subset of at least  $N$  iterates. Let  $\rho_k$  be a rigidity sequence for  $R_n$ . Choose  $\rho_1, \rho_2, \dots, \rho_N$  such that

$$(7) \quad \mu(R_n^{\rho_i} A \cap A) > (1 - \frac{\delta}{n})\mu(A)$$

for  $A \in P_n$ . Using a similar approximation as in [1], then a rigidity condition like (7) extends to all measurable sets  $A$  such that  $0 < \mu(A) < 1$ . Thus, (6) and (7) together can be used to show that the following set is infinite:

$$\{n \in \mathbb{N} : \mu(T^n A \cap A) - \mu(A)^2 > 0\}.$$



## 5. MIXING TOWERPLEX DETAILS

Partition  $X$  into two equal sets  $X_1$  and  $Y_1$  (i.e.  $\mu(X_1) = \mu(Y_1) = 1/2$ ). Initialize  $R_1$  isomorphic to  $R$  and  $S_1$  isomorphic to  $S$  to operate on  $X_1$  and  $Y_1$ , respectively. Define  $T_1(x) = R_1(x)$  for  $x \in X_1$  and  $T_1(x) = S_1(x)$  for  $x \in Y_1$ . Produce Rohklin towers of height  $h_1$  with residual less than  $\epsilon_1/2$  for each of  $R_1$  and  $S_1$ . In particular, let  $I_1, J_1$  be the base of the  $R_1$ -tower and  $S_1$ -tower such that  $\mu(\bigcup_{i=0}^{h_1-1} R_1^i(I_1)) > 1/2(1 - \epsilon_1)$  and  $\mu(\bigcup_{i=0}^{h_1-1} S_1^i(J_1)) > 1/2(1 - \epsilon_1)$ . Let  $X_1^* = X_1 \setminus \bigcup_{i=0}^{h_1-1} R_1^i(I_1)$  and  $Y_1^* = Y_1 \setminus \bigcup_{i=0}^{h_1-1} S_1^i(J_1)$  be the residuals for the  $R_1$  and  $S_1$  towers, respectively. Choose  $I'_1 \subset I_1$  and  $J'_1 \subset J_1$  such that

$$\mu(I'_1) = r_1 \mu(I_1) \text{ and } \mu(J'_1) = s_1 \mu(J_1).$$

Set  $X_2 = X_1 \setminus [\bigcup_{i=0}^{h_1-1} R_1^i(I'_1)] \cup [\bigcup_{i=0}^{h_1-1} S_1^i(J'_1)]$  and  $Y_2 = Y_1 \setminus [\bigcup_{i=0}^{h_1-1} S_1^i(J'_1)] \cup [\bigcup_{i=0}^{h_1-1} R_1^i(I'_1)]$ . We will define second stage transformations  $R_2 : X_2 \rightarrow X_2$  and  $S_2 : Y_2 \rightarrow Y_2$ . First, it may be necessary to add or subtract measure from the residuals so that  $X_2$  is scaled properly to define  $R_2$ , and  $Y_2$  is scaled properly to define  $S_2$ .

**5.1. Tower Rescaling.** In the case where  $\mu(I'_1) \neq \mu(J'_1)$ , we give a procedure for transferring measure between the towers and the residuals. This is done in order to consistently define  $R_2$  and  $S_2$  on the new inflated or deflated towers. Let  $a = \mu(\bigcup_{i=0}^{h_1-1} R_1^i(I_1))$  and  $b = h_1(\mu(J'_1) - \mu(I'_1))$ . Let  $c$  be the scaling factor and  $d$  the amount of measure transferred between  $\bigcup_{i=0}^{h_1-1} S_1^i(J'_1)$  and  $X_1^*$ . Thus,  $a + b - d = ca$  and  $1/2 - a + d = c(1/2 - a)$ . The goal is to solve two unknowns  $d$  and  $c$  in terms of the other values. Hence,  $d = (1 - 2a)b$  and  $c = 1 + 2b$ .

**5.1.1.  $R$  Rescaling.** If  $d > 0$ , define  $I_1^* \subset J'_1$  such that  $\mu(I_1^*) = d/h_1$ . Let  $X'_1 = X_1^* \cup (\bigcup_{i=0}^{h_1-1} R_1^i(I_1^*))$ . If  $d = 0$ , set  $X'_1 = X_1^*$ . If  $d < 0$ , transfer measure from  $X_1^*$  to the tower. Choose disjoint sets  $I_1^*(0), I_1^*(1), \dots, I_1^*(h_1 - 1)$  contained in  $X_1^*$  such that  $\mu(I_1^*(i)) = d/h_1$ . Denote  $I_1^* = I_1^*(0)$ . Begin by defining  $\mu$  measure preserving map  $\alpha_1$  such that  $I_1^*(i + 1) = \alpha_1(I_1^*(i))$  for  $i = 0, 1, \dots, h_1 - 2$ . In this case, let  $X'_1 = X_1^* \setminus [\bigcup_{i=0}^{h_1-1} I_1^*(i)]$ .

**5.1.2.  $S$  Rescaling.** The direction mass is transferred depends on the sign of  $b$  above. If  $d > 0$ , then  $\mu(J'_1) > \mu(I'_1)$  and mass is transferred from the residual  $Y_1^*$  to the  $S_1$ -tower. Choose disjoint sets  $J_1^*(0), J_1^*(1), \dots, J_1^*(h_1 - 1)$  contained in  $Y_1^*$  such that  $\mu(J_1^*(i)) = d/h_1$ . Denote  $J_1^* = J_1^*(0)$ . Begin by defining  $\mu$  measure preserving map  $\beta_1$  such that  $J_1^*(i + 1) = \beta_1(J_1^*(i))$  for  $i = 0, 1, \dots, h_1 - 2$ . In this case, let  $Y'_1 = Y_1^* \setminus [\bigcup_{i=0}^{h_1-1} J_1^*(i)]$ . If  $d = 0$ , set  $Y'_1 = Y_1^*$ . If  $d < 0$ , transfer measure from the  $S_1$ -tower to

the residual  $Y_1^*$ . Define  $J_1^* \subset J_1 \setminus J_1'$  such that  $\mu(J_1^*) = d/h_1$ . Let  $Y_1' = Y_1^* \cup (\bigcup_{i=0}^{h_1-1} S_1^i(J_1^*))$ .

Note, if  $d \neq 0$ , then both  $\epsilon_1$  and  $\mu(X_1^*)$  may be chosen small enough (relative to  $r_1$ ) to ensure the following solutions lead to well-defined sets and mappings. For subsequent stages, assume  $\epsilon_n$  is chosen small enough to force well-defined rescaling parameters, transfer sets and mappings  $R_n, S_n$ .

**5.2. Stage 2 Construction.** We have specified three cases:  $d > 0$ ,  $d = 0$  and  $d < 0$ . The case  $d = 0$ , can be handled along with the case  $d > 0$ . This gives two essential cases. Note the case  $d < 0$  is analogous to the case  $d > 0$ , with the roles of  $R_1$  and  $S_1$  reversed. However, due to a key distinction in the handling of the  $R$ -rescaling and the  $S$ -rescaling, it is important to clearly define  $R_2$  and  $S_2$  in both cases.

**Case 5.1** ( $d \geq 0$ ). Define  $\tau_1 : X_1' \rightarrow X_1^*$  as a measure preserving map between normalized spaces  $(X_1', \mathbb{B} \cap X_1', \frac{\mu}{\mu(X_1')})$  and  $(X_1^*, \mathbb{B} \cap X_1^*, \frac{\mu}{\mu(X_1^*)})$ . Extend  $\tau_1$  to the new tower base,

$$\tau_1 : [I_1 \setminus I_1'] \cup [J_1' \setminus I_1^*] \rightarrow I_1$$

such that  $\tau_1$  preserves normalized measure between

$$\frac{\mu}{\mu([I_1 \setminus I_1'] \cup [J_1' \setminus I_1^*])} \text{ and } \frac{\mu}{\mu(I_1)}.$$

Define  $\tau_1$  on the remainder of the tower consistently such that

$$\tau_1(x) = \begin{cases} R_1^i \circ \tau_1 \circ R_1^{-i}(x) & \text{if } x \in R_1^i(I_1 \setminus I_1') \text{ for } 0 \leq i < h_1 \\ R_1^i \circ \tau_1 \circ S_1^{-i}(x) & \text{if } x \in S_1^i(J_1' \setminus I_1^*) \text{ for } 0 \leq i < h_1 \end{cases}$$

Define  $R_2 : X_2 \rightarrow X_2$  as  $R_2 = \tau_1^{-1} \circ R_1 \circ \tau_1$ . Note

$$R_2(x) = \begin{cases} S_1(x) & \text{if } x \in S_1^i(J_1' \setminus I_1^*) \text{ for } 0 \leq i < h_1 - 1 \\ R_1(x) & \text{if } x \in R_1^i(I_1 \setminus I_1') \text{ for } 0 \leq i < h_1 - 1 \end{cases}$$

Clearly,  $R_2$  is isomorphic to  $R_1$  and  $R$ .

Define  $\psi_1 : Y_1' \rightarrow Y_1^*$  as a measure preserving map between normalized spaces  $(Y_1', \mathbb{B} \cap Y_1', \frac{\mu}{\mu(Y_1')})$  and  $(Y_1^*, \mathbb{B} \cap Y_1^*, \frac{\mu}{\mu(Y_1^*)})$ . Extend  $\psi_1$  to the new tower base,

$$\psi_1 : [J_1 \setminus J_1'] \cup J_1^* \cup I_1' \rightarrow J_1$$

such that  $\psi_1$  preserves normalized measure between

$$\frac{\mu}{\mu([J_1 \setminus J_1'] \cup J_1^* \cup I_1')} \text{ and } \frac{\mu}{\mu(J_1)}.$$

Define  $\psi_1$  on the remainder of the tower consistently such that

$$\psi_1(x) = \begin{cases} S_1^i \circ \psi_1 \circ S_1^{-i}(x) & \text{if } x \in S_1^i(J_1 \setminus J_1') \text{ for } 0 \leq i < h_1 \\ S_1^i \circ \psi_1 \circ R_1^{-i}(x) & \text{if } x \in R_1^i(I_1') \text{ for } 0 \leq i < h_1 \\ \beta_1^i \circ \psi_1 \circ \beta_1^{-i}(x) & \text{if } x \in J_1^*(i) \text{ for } 0 \leq i < h_1 \end{cases}$$

In this case, define  $S_2 : Y_2 \rightarrow Y_2$  such that  $S_2 = \psi_1^{-1} \circ S_1 \circ \psi_1$ . Note

$$S_2(x) = \begin{cases} R_1(x) & \text{if } x \in R_1^i I_1' \text{ for } 0 \leq i < h_1 - 1 \\ S_1(x) & \text{if } x \in S_1^i(J_1 \setminus J_1') \text{ for } 0 \leq i < h_1 - 1 \\ \beta_1(x) & \text{if } x \in J_1^*(i) \text{ for } 0 \leq i < h_1 - 1 \\ \psi_1^{-1} \circ S_1 \circ \psi_1(x) & \text{if } x \in Y_1' \cup S_1^{h_1-1}(J_1 \setminus J_1') \cup R_1^{h_1-1} I_1' \cup \beta_1^{h_1-1} J_1^* \end{cases}$$

and  $S_2$  is isomorphic to  $S_1$  and  $S$ .

**Case 5.2** ( $d < 0$ ). Define  $\tau_1 : X_1' \rightarrow X_1^*$  as a measure preserving map between normalized spaces  $(X_1', \mathbb{B} \cap X_1', \frac{\mu}{\mu(X_1')})$  and  $(X_1^*, \mathbb{B} \cap X_1^*, \frac{\mu}{\mu(X_1^*)})$ . Extend  $\tau_1$  to the new tower base,

$$\tau_1 : [I_1 \setminus I_1'] \cup I_1^* \cup J_1' \rightarrow I_1$$

such that  $\tau_1$  preserves normalized measure between

$$\frac{\mu}{\mu([I_1 \setminus I_1'] \cup I_1^* \cup J_1')} \text{ and } \frac{\mu}{\mu(I_1)}.$$

Define  $\tau_1$  on the remainder of the tower consistently such that

$$\tau_1(x) = \begin{cases} R_1^i \circ \tau_1 \circ R_1^{-i}(x) & \text{if } x \in R_1^i(I_1 \setminus I_1') \text{ for } 0 \leq i < h_1 \\ R_1^i \circ \tau_1 \circ S_1^{-i}(x) & \text{if } x \in S_1^i(J_1') \text{ for } 0 \leq i < h_1 \\ \alpha_1^i \circ \tau_1 \circ \alpha_1^{-i}(x) & \text{if } x \in I_1^*(i) \text{ for } 0 \leq i < h_1 \end{cases}$$

In this case, define  $R_2 : X_2 \rightarrow X_2$  such that

$$R_2(x) = \begin{cases} S_1(x) & \text{if } x \in S_1^i J_1' \text{ for } 0 \leq i < h_1 - 1 \\ R_1(x) & \text{if } x \in R_1^i(I_1 \setminus I_1') \text{ for } 0 \leq i < h_1 - 1 \\ \alpha_1(x) & \text{if } x \in I_1^*(i) \text{ for } 0 \leq i < h_1 - 1 \\ \tau_1^{-1} \circ R_1 \circ \tau_1(x) & \text{if } x \in X_1' \cup S_1^{h_1-1}(I_1 \setminus I_1') \cup S_1^{h_1-1} J_1' \cup \alpha_1^{h_1-1} I_1^* \end{cases}$$

Clearly,  $R_2$  is isomorphic to  $R_1$  and  $R$ .

Define  $\psi_1 : Y_1' \rightarrow Y_1^*$  as a measure preserving map between normalized spaces  $(Y_1', \mathbb{B} \cap Y_1', \frac{\mu}{\mu(Y_1')})$  and  $(Y_1^*, \mathbb{B} \cap Y_1^*, \frac{\mu}{\mu(Y_1^*)})$ . Extend  $\psi_1$  to the new tower base,

$$\psi_1 : [J_1 \setminus (J_1' \cup J_1^*)] \cup I_1' \rightarrow J_1$$

such that  $\psi_1$  preserves normalized measure between

$$\frac{\mu}{\mu([J_1 \setminus (J_1' \cup J_1^*)] \cup I_1')} \text{ and } \frac{\mu}{\mu(J_1)}.$$

Define  $\psi_1$  on the remainder of the tower consistently such that

$$\psi_1(x) = \begin{cases} S_1^i \circ \psi_1 \circ S_1^{-i}(x) & \text{if } x \in S_1^i(J_1 \setminus [J_1' \cup J_1^*]) \text{ for } 0 \leq i < h_1 \\ S_1^i \circ \psi_1 \circ R_1^{-i}(x) & \text{if } x \in R_1^i(I_1') \text{ for } 0 \leq i < h_1 \end{cases}$$

Define  $S_2 : Y_2 \rightarrow Y_2$  such that  $S_2 = \psi_1^{-1} \circ S_1 \circ \psi_1$ . Note

$$S_2(x) = \begin{cases} R_1(x) & \text{if } x \in R_1^i(I_1') \text{ for } 0 \leq i < h_1 - 1 \\ S_1(x) & \text{if } x \in S_1^i(J_1 \setminus [J_1' \cup J_1^*]) \text{ for } 0 \leq i < h_1 - 1 \end{cases}$$

*Transformation  $S_2$  is isomorphic to  $S_1$  and  $S$ .*

Define  $T_2$  as

$$T_2(x) = \begin{cases} R_2(x) & \text{if } x \in X_2 \\ S_2(x) & \text{if } x \in Y_2 \end{cases}$$

Clearly, neither  $T_1$  nor  $T_2$  are ergodic. For  $T_1$ ,  $X_1$  and  $Y_1$  are ergodic components, and  $X_2, Y_2$  are ergodic components for  $T_2$ . See the appendix for a pictorial of the multiplexing operation used to produce  $R_2$  and  $S_2$  from  $R_1, S_1$  and the intermediary maps defined in this section.

**5.3. General Multiplexing Operation.** For  $n \geq 1$ , suppose that  $R_n$  and  $S_n$  have been defined on  $X_n$  and  $Y_n$  respectively. Construct Rohklin towers of height  $h_n$  for each  $R_n$  and  $S_n$ , and such that  $I_n$  is the base of the  $R_n$  tower,  $J_n$  is the base of the  $S_n$  tower, and  $\mu(\bigcup_{i=0}^{h_n-1} R_n^i I_n) + \mu(\bigcup_{i=0}^{h_n-1} S_n^i J_n) > 1 - \epsilon_n$ . Let  $I'_n \subset I_n$  be such that  $\mu(I'_n) = r_n \mu(I_n)$ . Similarly, suppose  $J'_n \subset J_n$  such that  $\mu(J'_n) = s_n \mu(J_n)$ .

We define  $R_{n+1}$  and  $S_{n+1}$  by switching the subcolumns

$$\{I'_n, R_n(I'_n), R_n^2(I'_n), \dots, R_n^{h_n-1}(I'_n)\}$$

and

$$\{J'_n, S_n(J'_n), S_n^2(J'_n), \dots, S_n^{h_n-1}(J'_n)\}.$$

Let

$$\begin{aligned} X_{n+1} &= [\bigcup_{i=0}^{h_n-1} R_n^i(I_n \setminus I'_n)] \cup [\bigcup_{i=0}^{h_n-1} S_n^i J'_n] \cup [X_n \setminus \bigcup_{i=0}^{h_n-1} R_n^i I_n] \\ Y_{n+1} &= [\bigcup_{i=0}^{h_n-1} S_n^i(J_n \setminus J'_n)] \cup [\bigcup_{i=0}^{h_n-1} R_n^i I'_n] \cup [Y_n \setminus \bigcup_{i=0}^{h_n-1} S_n^i J_n]. \end{aligned}$$

As in the initial case, it may be necessary to transfer measure between each column and its respective residual. We can follow the same algorithm as above, and define maps  $\tau_n, \alpha_n, \psi_n$  and  $\beta_n$ . Thus, we get the following definitions:

**Case 5.3** ( $d \geq 0$ ).

$$\tau_n(x) = \begin{cases} R_n^i \circ \tau_n \circ R_n^{-i}(x) & \text{if } x \in R_n^i(I_n \setminus I'_n) \text{ for } 0 \leq i < h_n \\ R_n^i \circ \tau_n \circ S_n^{-i}(x) & \text{if } x \in S_n^i(J'_n \setminus I_1^*) \text{ for } 0 \leq i < h_n \end{cases}$$

$$R_{n+1}(x) = \begin{cases} S_n(x) & \text{if } x \in S_n^i(J'_n \setminus I_n^*) \text{ for } 0 \leq i < h_n - 1 \\ R_n(x) & \text{if } x \in R_n^i(I_n \setminus I'_n) \text{ for } 0 \leq i < h_n - 1 \\ \tau_n^{-1} \circ R_n \circ \tau_n(x) & \text{if } x \in X'_n \cup R_n^{h_n-1}(I_n \setminus I'_n) \cup S_n^{h_n-1}(J'_n \setminus I_n^*) \end{cases}$$

and  $R_{n+1} = \tau_n^{-1} \circ R_n \circ \tau_n$ .

$$\psi_n(x) = \begin{cases} S_n^i \circ \psi_n \circ S_n^{-i}(x) & \text{if } x \in S_n^i(J_n \setminus J'_n) \text{ for } 0 \leq i < h_n \\ S_n^i \circ \psi_n \circ R_n^{-i}(x) & \text{if } x \in R_n^i(I'_n) \text{ for } 0 \leq i < h_n \\ \beta_n^i \circ \psi_n \circ \beta_n^{-i}(x) & \text{if } x \in J_n^*(i) \text{ for } 0 \leq i < h_n \end{cases}$$

$$S_{n+1}(x) = \begin{cases} R_n(x) & \text{if } x \in R_n^i I'_n \text{ for } 0 \leq i < h_n - 1 \\ S_n(x) & \text{if } x \in S_n^i(J_n \setminus J'_n) \text{ for } 0 \leq i < h_n - 1 \\ \beta_n(x) & \text{if } x \in J_n^*(i) \text{ for } 0 \leq i < h_n - 1 \\ \psi_n^{-1} \circ S_n \circ \psi_n(x) & \text{if } x \in Y'_n \cup S_n^{h_n-1}(J_n \setminus J'_n) \cup R_n^{h_n-1} I'_n \cup \beta_n^{h_n-1} J_n^* \end{cases}$$

and  $S_{n+1} = \psi_n^{-1} \circ S_n \circ \psi_n$ .

**Case 5.4** ( $d < 0$ ).

$$\tau_n(x) = \begin{cases} R_n^i \circ \tau_n \circ R_n^{-i}(x) & \text{if } x \in R_n^i(I_n \setminus I'_n) \text{ for } 0 \leq i < h_n \\ R_n^i \circ \tau_n \circ S_n^{-i}(x) & \text{if } x \in S_n^i(J'_n) \text{ for } 0 \leq i < h_n \\ \alpha_n^i \circ \tau_n \circ \alpha_n^{-i}(x) & \text{if } x \in I_n^*(i) \text{ for } 0 \leq i < h_n \end{cases}$$

$$R_{n+1}(x) = \begin{cases} S_n(x) & \text{if } x \in S_n^i J'_n \text{ for } 0 \leq i < h_n - 1 \\ R_n(x) & \text{if } x \in R_n^i(I_n \setminus I'_n) \text{ for } 0 \leq i < h_n - 1 \\ \alpha_n(x) & \text{if } x \in I_n^*(i) \text{ for } 0 \leq i < h_n - 1 \\ \tau_n^{-1} \circ R_n \circ \tau_n(x) & \text{if } x \in X'_n \cup R_n^{h_n-1}(I_n \setminus I'_n) \cup S_n^{h_n-1} J'_n \cup \alpha_n^{h_n-1} I_n^* \end{cases}$$

and  $R_{n+1} = \tau_n^{-1} \circ R_n \circ \tau_n$ .

$$\psi_n(x) = \begin{cases} S_n^i \circ \psi_n \circ S_n^{-i}(x) & \text{if } x \in S_n^i(J_n \setminus [J'_n \cup J_n^*]) \text{ for } 0 \leq i < h_n \\ S_n^i \circ \psi_n \circ R_n^{-i}(x) & \text{if } x \in R_n^i(I'_n) \text{ for } 0 \leq i < h_n \end{cases}$$

$$S_{n+1}(x) = \begin{cases} R_n(x) & \text{if } x \in R_n^i(I'_n) \text{ for } 0 \leq i < h_n - 1 \\ S_n(x) & \text{if } x \in S_n^i(J_n \setminus [J'_n \cup J_n^*]) \text{ for } 0 \leq i < h_n - 1 \\ \psi_n^{-1} \circ S_n \circ \psi_n(x) & \text{if } x \in Y'_n \cup S_n^{h_n-1}(J_n \setminus [J'_n \cup J_n^*]) \cup R_n^{h_n-1}(I'_n) \end{cases}$$

and  $S_{n+1} = \psi_n^{-1} \circ S_n \circ \psi_n$ .

**5.4. The Limiting Transformation.** Define the transformation  $T_{n+1} : X_{n+1} \cup Y_{n+1} \rightarrow X_{n+1} \cup Y_{n+1}$  such that

$$T_{n+1}(x) = \begin{cases} R_{n+1}(x) & \text{if } x \in X_{n+1} \\ S_{n+1}(x) & \text{if } x \in Y_{n+1} \end{cases}$$

The set where  $T_{n+1} \neq T_n$  is determined by the top levels of the Rokhlin towers, the residuals and the transfer sets. Note the transfer set has measure  $d$ . Since this set is used to adjust the size of the residuals between stages, it can be bounded below a constant multiple of  $\epsilon_n$ . Thus, there is a fixed constant  $\kappa$ , independent of  $n$ , such that  $T_{n+1}(x) = T_n(x)$  except for  $x$  in a set of measure less than  $\kappa(\epsilon_n + 1/h_n)$ . Since  $\sum_{n=1}^{\infty} (\epsilon_n + 1/h_n) < \infty$ ,  $T(x) = \lim_{n \rightarrow \infty} T_n(x)$  exists almost everywhere, and preserves normalized

Lebesgue measure. Without loss of generality, we may assume  $\kappa$  and  $h_n$  are chosen such that if

$$E_n = \{x \in X | T_{n+1}(x) \neq T_n(x)\}$$

then  $\mu(E_n) < \kappa \epsilon_n$  for  $n \in \mathbb{N}$ . In the following section, additional structure and conditions are implemented to ensure that  $T$  inherits properties from  $R$  and  $S$ , and is also ergodic.

For the remainder of this paper, assume the parameters are chosen such that

- (1)  $\lim_{n \rightarrow \infty} s_n = 0$ ;
- (2)  $\sum_{n=1}^{\infty} r_n = \sum_{n=1}^{\infty} s_n = \infty$ ;
- (3)  $\lim_{n \rightarrow \infty} \mu(X_n) = 0$ ;
- (4)  $\sum_{n=1}^{\infty} \epsilon_n < \infty$ .

**5.5. Isomorphism Chain Consistency.** Suppose  $S$  is a strong mixing transformation on  $(Y, \mathcal{B}, \mu)$ . We will use the multiplexing procedure defined in the previous section to produce a "slow" mixing transformation  $T$ . Let  $\mu_n$  be normalized Lebesgue probability measure on  $Y_n$ . i.e.  $\mu_n = \mu/\mu(Y_n)$ .

For  $n \in \mathbb{N}$ , let  $P_n$  be a refining sequence of finite partitions which generates the sigma algebra. By refining  $P_n$  further if necessary, assume  $X_n, Y_n, X_n^*, Y_n^* \in P_n$ . Also, assume  $R_n^i(I_n'), R_n^i(I_n \setminus I_n'), S_n^i(J_n'), S_n^i(J_n \setminus J_n')$  are elements of  $P_n$  for  $0 \leq i < h_n$ . Finally, assume for  $0 \leq i < h_n - 1$ , if  $p \in P_n$  and  $p \subset R_n^i(I_n)$  then  $R_n(p) \in P_n$ . Likewise, assume for  $0 \leq i < h_n - 1$ , if  $p \in P_n$  and  $p \subset S_n^i(J_n)$  then  $S_n(p) \in P_n$ . Previously, we required that  $\psi_n$  map certain finite orbits from the  $S_n$  and  $R_n$  towers to a corresponding orbit in the  $S_{n+1}$  tower. In this section, further regularity is imposed on  $\psi_n$  relative to  $P_n$  to ensure dynamical properties of  $S_n$  are inherited by  $S_{n+1}$ .

Let  $P'_n = \{p \in P_n | p \subset \bigcup_{i=0}^{h_n-1} S_n^i(J_n \setminus J_n')\}$ . For each of the following three cases, impose the corresponding restriction on  $\psi_n$ :

- (1) for  $d = 0$  and  $p \in P'_n$ ,  $\psi_n$  is the identity map (i.e.  $\psi_n(p) = p$ );
- (2) for  $d > 0$  and  $p \in P'_n$ ,  $\psi_n(p) \subset p$ ;
- (3) for  $d < 0$  and  $p \in P'_n$ ,  $p \subset \psi_n(p)$ .

This can be accomplished by uniformly distributing the appropriate mass from the sets  $S_n^i(J_n^*)$  using  $\psi_n$ . Note that  $\psi_n$  either preserves Lebesgue measure in the case  $d = 0$ , or  $\psi_n$  contracts sets relative to Lebesgue measure in the case  $d > 0$ , or it inflates measure in the case  $d < 0$ . In all three cases, for  $p \in P'_n$ ,

$$\frac{\mu(p)}{\mu(\psi_n(p))} = \frac{\mu(Y_{n+1})}{\mu(Y_n)}.$$

It is straightforward to verify for any set  $A$  measurable relative to  $P'_n$ ,

$$\mu(A \triangle \psi_n A) < \left| \frac{\mu(Y_{n+1})}{\mu(Y_n)} - 1 \right|.$$

The properties of  $\psi_n$  allow approximation of  $S_{n+1}$  by  $S_n$  indefinitely over time. This is needed to establish mixing for the limiting transformation  $T$ .

Since each  $S_n$  is strongly mixing on  $Y_n$ , then for all  $A, B \in P'_n$ ,

$$\lim_{i \rightarrow \infty} \mu_n(A \cap S_n^i B) = \mu_n(A) \mu_n(B).$$

Prior to establishing strong mixing, we prove a lemma which is part of a similar lemma shown in [1]. For  $p \in P'_n$ ,

$$\frac{\mu(p)}{\mu(\psi_n(p))} = \frac{\mu(Y_{n+1})}{\mu(Y_n)}.$$

It is straightforward to verify for any set  $A$  measurable relative to  $P'_n$ ,

$$\begin{aligned} \mu(A \triangle \psi_n A) &= \mu(A) - \mu(\psi_n(A)) \\ &\leq \mu(\psi_n A) \left[ \frac{\mu(Y_{n+1})}{\mu(Y_n)} - 1 \right] = \frac{\mu(\psi_n A)}{\mu(Y_n)} [\mu(Y_{n+1}) - \mu(Y_n)]. \end{aligned}$$

and for any measurable set  $C \subset Y_n$ ,

$$|\mu(\psi_n^{-1} C) - \mu(C)| < \left| \frac{\mu(Y_{n+1})}{\mu(Y_n)} - 1 \right|.$$

These two properties are used in the following lemma to show  $S_{n+1}$  inherits dynamical properties from  $S_n$  indefinitely over time. Let  $Q_n = \{\psi_n(p) : p \in P'_n\}$ .

**Lemma 5.5.** *Suppose  $\delta > 0$  and  $n \in \mathbb{N}$  is chosen such that*

$$\epsilon_n + \mu(X_n) < \frac{\delta}{6}.$$

*Then for  $A, B \in Q_n$  and  $i \in \mathbb{N}$ ,*

$$|\mu(S_{n+1}^i A \cap B) - \mu(A) \mu(B)| < |\mu(S_n^i A \cap B) - \mu(A) \mu(B)| + \delta.$$

*Proof.* For  $A, B \in Q_n$ , let  $A' = \psi_n^{-1} A$  and  $B' = \psi_n^{-1} B$ . Thus,  $\mu(A' \triangle A) = \mu(\psi_n^{-1}(A \setminus \psi_n A)) < \delta/6$  and  $\mu(B' \triangle B) < \delta/6$ . By applying the triangle

inequality several times, we get the following approximation:

$$\begin{aligned}
|\mu(S_{n+1}^i A \cap B) - \mu(S_n^i A \cap B)| & \\
&\leq |\mu(S_{n+1}^i A' \cap B') - \mu(S_n^i A \cap B)| + \frac{\delta}{3} \\
&= |\mu(\psi_n^{-1} S_n^i \psi_n A' \cap B') - \mu(S_n^i A \cap B)| + \frac{\delta}{3} \\
&= |\mu(\psi_n^{-1} (S_n^i \psi_n A' \cap \psi_n B')) - \mu(S_n^i A \cap B)| + \frac{\delta}{3} \\
&= |\mu(\psi_n^{-1} (S_n^i A \cap B)) - \mu(S_n^i A \cap B)| + \frac{\delta}{3} \\
&< \frac{\delta}{2}.
\end{aligned}$$

Similarly,

$$|\mu(S_{n+1}^i F \cap F) - \mu(S_n^i F \cap F)| < \frac{\delta}{2}.$$

Therefore,

$$|\mu(S_{n+1}^i A \cap B) - \mu(A)\mu(B)| < |\mu(S_n^i A \cap B) - \mu(A)\mu(B)| + \delta.$$

□

## 6. TOWERPLEXES WITH SINGULAR SPECTRUM

If  $\Phi$  is the space of ergodic measure preserving transformations on a separable probability space, then the tower multiplexing operation defines a mapping

$$\mathcal{M} : \Phi \times \Phi \rightarrow \Phi.$$

The mapping also depends on a collection of parameters  $\mathcal{P}$ . Thus, we may write  $T = \mathcal{M}(R, S, \mathcal{P})$  to represent the multiplexed transformation  $T$  produced from transformations  $R$  and  $S$ . In [1], the transformation  $R$  is ergodic and rigid, and  $S$  is weak mixing. In particular,  $S$  is set to the Chacon3 transformation, and the parameters are defined such that  $S$  is a "dissipating" component. Given  $R$  ergodic with rigidity sequence  $(\rho_n)_{n=1}^\infty$ , it is shown there exists  $\mathcal{P}$  such that  $T = \mathcal{M}(R, S, \mathcal{P})$  is weak mixing with rigidity sequence  $\rho_n$ .

In this paper, we use the tower multiplexing technique to produce a transformation  $T = \mathcal{M}(S, R, \mathcal{P})$  with continuous singular spectrum. Again, the second component transformation will be a dissipating component. However, we flip the roles of  $R$  and  $S$ , so  $R$  (rigidity) is used in the second component. The parameter collection  $\mathcal{P}$  includes sequences  $r_n$  and  $s_n$ . As in [1], associate  $r_n$  with  $R$  and  $s_n$  with  $S$ . Associate  $X_n$  with the second component transformation  $R$ , and  $Y_n$  with  $S$ . Other parameters included in  $\mathcal{P}$  are  $\epsilon_n$  and  $h_n$ . We have the following main theorem.



**Theorem 6.1.** *Let  $S$  be an invertible ergodic measure preserving transformation with weak limit  $S^{m_n} \rightarrow S_0$  as  $n \rightarrow \infty$ . There exist a rigid weak mixing transformation  $R$ , and parameter  $\mathcal{P}$  such that*

$$T = \mathcal{M}(S, R, \mathcal{P})$$

*is weak mixing with singular spectrum, and  $T^{m_n} \rightarrow S_0$ .*

Prior to sketching a proof to the previous theorem, we will use the following result from [5].

**Proposition 6.2** (B. Fayad). *Let  $(X, \mathcal{B}, \mu, T)$  be an invertible ergodic measure preserving system. If for any complex nonzero, mean zero function  $f \in L^2(X, \mu)$ , there exists a measurable set  $E \subset X$  with  $\mu(E) > 0$ , and a strictly increasing sequence  $\ell_n$ , such that for every  $x \in E$ , we have*

$$\limsup_{n \rightarrow \infty} \frac{1}{n} \left| \sum_{i=0}^{n-1} f(T^{\ell_i} x) \right| > 0,$$

*then the maximal spectral type of the unitary operator associated to  $(X, \mathcal{B}, \mu, T)$  is singular.*

**Proof of Theorem 6.1:** Let  $S$  be an invertible ergodic measure preserving transformation with limit  $S_0 = w^* - \lim_{n \rightarrow \infty} S^{m_n}$ . We can define a rigid weak mixing transformation  $R$  such that the dissipating component  $R$  in the multiplexed transformation  $T = \mathcal{M}(S, R, \mathcal{P})$  will allow  $T$  to satisfy the previous proposition. We still require that parameters  $r_n$  and  $s_n$  have the same properties as in [1] except with roles reversed. In particular,  $r_n$  for  $R$  satisfies  $r_n = 1/2$ , and  $s_n = 1/2(n+2)$ . This ensures that the base of  $R_n, X_n$ , satisfies  $\lim_{n \rightarrow \infty} \mu(X_n) = 0$  and  $\sum_{n=1}^{\infty} \mu(X_n) = \infty$  with the  $X_n$  approximately independent. The same technique to establish the rigidity sequence in [1] can be used to establish weak convergence to  $S_0$  along  $m_n$ . Ergodicity and weak mixing may be established in a similar manner as in [1]. Singular spectrum is established using the previous proposition and the fact that almost every point falls in a subset of  $X_n$  infinitely often. The transformation  $R$  is defined such that  $\ell_i$  are "strong" rigid times, and rigid multiples of rigid times.  $\square$

**Corollary 6.3.** *Suppose  $S$  is an invertible strong mixing transformation. There exist an invertible rigid transformation  $R$ , and parameter  $\mathcal{P}$  such that*

$$T = \mathcal{M}(S, R, \mathcal{P})$$

*is strong mixing with singular spectrum.*

**Corollary 6.4.** *Suppose  $S$  is an invertible ergodic measure preserving transformation, and  $(m_n)_{n=1}^{\infty}$  is a sequence such that the weak closure of  $\{S^{m_n} :$*

$n \in \mathbb{N}\}$  contains a countable set of limit points  $\{S_k : k \in \mathbb{N}\}$ . Then there exists a weak mixing transformation  $T$  with singular spectrum such that the weak closure of  $\{T^{m_n} : n \in \mathbb{N}\}$  contains the same countable set of limit points  $\{S_k : k \in \mathbb{N}\}$ .

**6.1. Multiple mixing towerplexes.** In [7], it is shown that any mixing transformation with singular spectrum is mixing of all orders. This implies the following corollary.

**Corollary 6.5.** *Given any strong mixing transformation  $S$ , there exist a rigid transformation  $R$ , and parameter  $\mathcal{P}$  such that*

$$T = \mathcal{M}(S, R, \mathcal{P})$$

*is mixing of all orders.*

**Question:** Given a mixing transformation  $S$ , is it possible to construct a rigid transformation  $R$  and parameter  $\mathcal{P}$  such that  $T = \mathcal{M}(S, R, \mathcal{P})$  has the same higher order mixing properties as  $S$  and has singular spectrum? This would be sufficient to prove that strong mixing implies mixing of all orders.

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